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On electromagnetic radiation from an accelerated charge

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Abstract. This paper is an attempt to set up a local criterion for the existence of electromagnetic radiation at a given spatial point, especially in view of the controversies in the case of Born fields of a uniformly accelerated charge. The criterion is that there cannot exist any time-like line passing through that point over which the magnetic field vanishes.

1. Introduction

While it is generally believed that an accelerated charge always has a radiation field associated with it, the case of a uniformly accelerated charge has led to conflicting conclusions (Pauli 1958, Fulton and Rohrlich 1960, Rohrlich 1965, Bondi and Gold 1955). The retarded fields for such a charge having the trajectory $z = +(\alpha^2 + t^2)^{1/2}$ ($c = 1$) as obtained by Born (1909) may be written as (see Rohrlich 1965):

$$\begin{aligned} &E_\rho = 8e\alpha^2\rho z/\xi^3 \qquad E_\phi = 0 \\ &E_z = -(4e\alpha^2/\xi^3)(\alpha^2 + t^2 - \rho^2 - z^2) \\ &B_\rho = B_z = 0 \qquad B_\phi = 8e\alpha^2\rho t/\xi^3 \end{aligned} \tag{1}$$

where we have used cylindrical coordinates (ρ, ϕ, z, t) and

$$\xi = [4\alpha^2\rho^2 + (\alpha^2 + t^2 - \rho^2 - z^2)^2]^{1/2}.$$

As $\boldsymbol{B} = 0$ for $t = 0$, it was argued that at that instant there was no radiation flux anywhere. Furthermore, the situation could be obtained at any other instant by a Lorentz transformation and hence apparently the radiation field is absent at all stages.

This reasoning has been criticised from two sides. Bondi and Gold (1955) argued that the solution obtained by Born was not correct as it gave a non-vanishing field even in the region $z + t < 0$. Rohrlich (1965), however, accepted the Born solution as correct and applied several other criteria and found a non-vanishing radiation flux.

In the present paper, the field due to a charge in the motion $z = +(\alpha^2 + \beta t^2)^{1/2}$ is presented. The field due to such a charge does not suffer from the defects pointed out by Bondi and Gold; nevertheless, there is a space-like hypersurface over which $\boldsymbol{B} = 0$. However, we think that this cannot be taken to mean an absence of radiation; for an observer, to see $\boldsymbol{B} = 0$ continuously, has to travel along a space-like line. For any time-like trajectory, $\boldsymbol{B}$ and hence $\boldsymbol{E} \times \boldsymbol{B}$ vanishes only for an instant.

This paper also makes a critical examination of the problem of radiation. As is well known, the Poynting vector may be made to vanish at any space–time point by a suitable Lorentz transformation if the field is non-null (see Misner *et al* 1973). It can also be seen that the field due to any charge distribution (accelerated or not) is necessarily non-null.

The usual practice of going over to a surface integral at infinity to determine whether there is net emission of radiation from the charge suffers from a formal inadequacy. A non-vanishing surface integral $\oint(\boldsymbol{E}\times\boldsymbol{B})\,.\,\mathrm{d}\boldsymbol{s}$ means as well that the volume integral $\oint(E^2+B^2)\,\mathrm{d}v$ diverges (since $|E|\sim|B|\sim\mathrm{O}(1/R)$) when the volume of integration is taken to be arbitrarily large.

Moreover, since the Poynting vector cannot be consistently interpreted as an energy flux, it remains a problem to answer unequivocally the question as to whether there is radiation at a particular spatial point. The 'local' criteria as discussed by Rohrlich (1965) require a knowledge of the state of motion of the sources and provide no answer to the problem as to how an observer is to decide whether there is radiation in his locale by performing purely local measurements.

The criterion that is suggested in the present work is to see whether there is any time-like line over which $\boldsymbol{B}$ vanishes. In more physical language this means we have to investigate whether an observer in physical motion can exist such that for him $\boldsymbol{B}$, and hence $\boldsymbol{E}\times\boldsymbol{B}$, is zero permanently or at least for a finite length of time. However, the existence of such a time-like line at some spatial point does not preclude the existence of radiation as a whole, since such time-like lines may, in general, not be possible through all spatial points.

2. Accelerated charge and Born fields

The fields due to a charge moving with arbitrary acceleration are well known. If $\boldsymbol{r}$ denotes the field point and $\boldsymbol{r}'$ the source point, and if t, t' denote the observation and retarded times, respectively,

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{e}{s^3}\{(1-u^2)(\boldsymbol{R}-R\boldsymbol{u})+\boldsymbol{R}\times[(\boldsymbol{R}-R\boldsymbol{u})\times\boldsymbol{a}]\}$$

$$\boldsymbol{B}(\boldsymbol{r},t)=\frac{\boldsymbol{R}\times\boldsymbol{E}}{R} \tag{2}$$

where $\boldsymbol{R}=\boldsymbol{r}(t)-\boldsymbol{r}'(t')$, $s=R-\boldsymbol{u}\,.\,\boldsymbol{R}$ and $\boldsymbol{u}$, $\boldsymbol{a}$ are the velocity and acceleration of the charge at the retarded time.

Conventionally, the first term in both $\boldsymbol{E}$ and $\boldsymbol{B}$ is said to be of the order of $1/R^2$ while the second terms are of the order of $1/R$. Then, as the determination of radiation involves an integration over a sphere at infinity, the first terms fall off faster and hence only the second terms contribute. Therefore these $1/R$ terms are called the radiation terms.

This splitting is, however, not permissible in general. Both $\boldsymbol{u}$ and $\boldsymbol{a}$ are to be evaluated at the retarded time $t'=t-R$ and they thus depend on t' and hence on R. Thus the R dependence of terms in $\boldsymbol{E}$ and $\boldsymbol{B}$ depends on the nature of the motion of the source. For hyperbolic motion:

$$z'=+(\alpha^2+t'^2)^{1/2}$$

$$\boldsymbol{u}(t')=\frac{t'\hat{z}}{(\alpha^2+t'^2)^{1/2}}\qquad \boldsymbol{a}(t')=\frac{\alpha^2}{(\alpha^2+t'^2)^{3/2}}\hat{z}$$

and one finds that both terms in $\boldsymbol{E}$ and $\boldsymbol{B}$ are of the same order in R. This is also evident from the Born solutions.

It may be noted, however, that if we transform to a frame in which $\boldsymbol{u}(t')=0$, only the second terms finally contribute to the surface integral of $\boldsymbol{E}\times\boldsymbol{B}$ over any finite sphere and hence determine radiation.

3. Hyperbolic motion with non-uniform acceleration

Let us consider the retarded fields due to a charge in the motion:

$$z=+(\alpha^2+\beta t^2)^{1/2} \tag{3}$$

where $0<\beta<1$. Let (x, y, z, t) denote the field point, and z_{ret} denote the source point. Then the fields are:

$$\begin{aligned}
E_x &= \frac{ex}{s^3}\left(1-\beta+\frac{z\alpha^2\beta}{z^3_{\text{ret}}}\right) \qquad E_y=\frac{ey}{s^3}\left(1-\beta+\frac{z\alpha^2\beta}{z^3_{\text{ret}}}\right)\\
E_z &= \frac{e}{s^3 z^3_{\text{ret}}}\{(z-z_{\text{ret}})[(1-\beta)z^3_{\text{ret}}+\alpha^2\beta z]-R[\alpha^2\beta t+\beta^2(1-\beta)(t-R)^3]\}\\
B_x &= \frac{-ey}{s^3 z^3_{\text{ret}}}[\alpha^2\beta t+\beta^2(1-\beta)(t-R)^3]\\
B_y &= \frac{ex}{s^3 z^3_{\text{ret}}}[\alpha^2\beta t+\beta^2(1-\beta)(t-R)^3]\\
B_z &= 0.
\end{aligned} \tag{4}$$

We have retained z_{ret} and R as, in general, explicit expressions in terms of (x, y, z, t) are not readily obtainable.

It may be easily verified that the fields (4) reduce to the Born fields when $\beta=1$.

The motion (3) is also hyperbolic, but has asymptotes along $z=\pm\sqrt{\beta}t$ instead of $z=\pm t$ as in the Born case. (It may be noted that this motion does not have the same invariance properties as that of uniform acceleration.) Thus, for $\beta\neq 1$ there is no region of space–time inaccessible to the charge and hence there is no region where the fields must vanish. Thus the fields (4) are free from the objections raised by Bondi and Gold. However, motion with $\beta=1$ involves $\dot{z}=1$ as $|t|\to\infty$ and hence the Born solutions with its apparent defects belong to a physically unattainable situation. We therefore suggest that the limit $\beta\to 1$ is not physically permissible.

4. The criterion for radiation

For fields of an accelerated charge, $\boldsymbol{E}\cdot\boldsymbol{B}=0$ and hence $\boldsymbol{E}\times\boldsymbol{B}=0$ implies $\boldsymbol{B}=0$.

We consider the point $\boldsymbol{r}$. To know whether there is any radiation at this point, we look at the equation $\boldsymbol{B}(\boldsymbol{r}, t)=0$. Depending on the motion of the source, this may represent a line, a two-dimensional surface, a three-dimensional surface, or may even be non-existent.

If $\boldsymbol{B}(\boldsymbol{r}, t) = 0$ is satisfied over a time-like line then for the observer whose world line is this line, $\boldsymbol{B} = 0$ always and hence there is no radiation for him at that point. Hence there is no real radiation field at that point.

When no such time-like line exists, a frame in which $\boldsymbol{B} = 0$ always cannot be found. Thus there must be radiation at that point.

This radiation criterion is strictly local and covariant, and tells us whether there is radiation at a particular spatial point by making purely local measurements. The application of this criterion for a charge moving with constant velocity is trivial, since a single Lorentz transformation to the rest frame of the charge makes $\boldsymbol{B} = 0$ everywhere and for all times, so that radiation is evidently absent.

For a charge moving in an arbitrary fashion in the absence of external electromagnetic radiation there may be some points or regions of space through which time-like lines over which $\boldsymbol{B} = 0$ exist. Evidently, there is no radiation at these points or regions. However, there may be other spatial points through which such $\boldsymbol{B} = 0$ time-like lines are not possible. Therefore at these latter points there is radiation.

Now we consider the problem as to whether the presence of radiation at a point may be causally related to some source. In the case under discussion, there is only a single point charge moving in some trajectory and no external radiation field. It is, therefore, natural to assign the radiation at any point to this charge, and we conclude that the charge itself is emitting radiation.

However, it may appear, in view of the conventional procedure of integrating the flux over a sphere enclosing the charge, that when such an integration is carried out the effect of radiation at one portion of the surface may be annulled by an opposite flux in some other part. This is clearly not possible since this would require the presence of an incoming wave and hence advanced fields, which have not been accounted for in the evaluation of $\boldsymbol{E}$ and $\boldsymbol{B}$. Thus, while radiation is absent as a whole only when $\boldsymbol{B} = 0$ time-like lines are possible through *all* points in space, absence of such a time-like line through any point implies the presence of radiation in the field as a whole.

The above criterion leads to the expected results in the standard case. For example, in the case of a charge moving uniformly in the circle:

$$x' = a \cos wt' \qquad y' = a \sin wt' \qquad z' = 0$$

the condition $\boldsymbol{B} = 0$ implies

$$z' = 0$$

or

$$x \cos w(t-R) + y \sin w(t-R) = 1/w^2 a.$$

This equation has real solutions for $\cos w(t-R)$ provided

$$x^2 + y^2 \geqslant 1/w^4 a^2 = a^2/u^4 > a^2$$

where we have used the fact that $u = aw$, and the velocity of the charge is always less than unity.

Thus, inside the circle $z \neq 0$, $x^2 + y^2 = a^2$ there is no point for which $\boldsymbol{B} = 0$ for any time. Evidently, radiation is present in this region; and by our preceding arguments, the charge in question radiates.

We consider finally the motion

$$z = +(\alpha^2 + \beta t^2)^{1/2}.$$

The condition $\boldsymbol{B}=0$ implies

$$\boldsymbol{r}=z\hat{z}$$

or

$$F=t-\frac{(R-t)^3\beta(1-\beta)}{\alpha^2}=0. \tag{5}$$

Thus, while there is no radiation at any point on the z axis (the z axis itself being a line along which any physical observer can move), for all other points $\boldsymbol{B}=0$ over the hypersurface defined by equation (5). However, as

$$\left(\frac{\partial F}{\partial t}\right)^2-\left(\frac{\partial F}{\partial x}\right)^2-\left(\frac{\partial F}{\partial y}\right)^2-\left(\frac{\partial F}{\partial z}\right)^2=1+\frac{6(R-t)^2\beta(1-\beta)}{\alpha^2}>0$$

this hypersurface is space-like for all $\boldsymbol{r}$ (except $\boldsymbol{r}=\boldsymbol{z}$). Consequently there is radiation at all such points, and the charge radiates.

For hyperbolic motion, with uniform acceleration, $\beta=1$ and the $\boldsymbol{B}=0$ hypersurface is $t=0$. This surface is space-like and hence there is no time-like line over which $\boldsymbol{B}=0$, except for points on the z axis. Consequently this charge also has a radiation field.

5. Fields in the rest frame

A transformation which reduces a charge moving along the trajectory $z=+(\alpha^2+\beta t^2)^{1/2}$ to rest has the form:

$$z'=f(z^2-\beta t^2).$$

The metric may be diagonalised by a transformation:

$$t'=\phi(tz^{-1/\beta})$$

where f and ϕ are arbitrary functions.

In the case of uniform acceleration, the metric in this co-moving non-inertial frame may be chosen to be static by choosing (see Moller 1957)

$$\begin{aligned} &x=x' \qquad y=y' \\ &z=(z'+\alpha)\cosh(t'/\alpha) \\ &t=(z'+\alpha)\sinh(t'/\alpha). \end{aligned} \tag{6}$$

Then direct transformation of the fields (or potentials) yields

$$\begin{aligned} &F'_{12}=F'_{23}=F'_{31}=0 \qquad \text{i.e.} \qquad \boldsymbol{B}'=0 \\ &F'_{14}=\frac{2x'e}{\xi\alpha}\left(1-\frac{c^2}{\xi^2}\right) \qquad F'_{24}=\frac{2y'e}{\xi\alpha}\left(1-\frac{c^2}{\xi^2}\right) \\ &F'_{34}=\frac{4(z'+\alpha)\alpha e}{\xi^3}[\rho^2+\alpha^2-(z'+\alpha)^2] \end{aligned} \tag{7}$$

where ξ is as in equation (1) and

$$c=\rho^2+\alpha^2+(z'+\alpha)^2.$$

Since $\boldsymbol{B}' = 0$ in this frame always, there seems to be no radiation in this frame. This does not contradict our conclusions in § 4 since this frame is non-inertial and the metric

$$ds^2 = (1 + z'/\alpha)^2 \, dt'^2 - dx'^2 - dy'^2 - dz'^2 \tag{8}$$

does not tend to Minkowski form even asymptotically.

Furthermore the metric has singular behaviour at the point $z' = -\alpha$. This point behaves as an event horizon at which the proper time interval vanishes. The fields also vanish at this point, but this is a phenomenon peculiar to this particular frame.

For $\beta \neq 1$ the metric cannot be chosen to be static for any choice of the functions f and ϕ, and in general the magnetic field does not vanish.

Here we come across a peculiar characteristic of uniform acceleration whose implications, if any, in the equivalence principle are not very clear.

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